

A SHORT REMARK ON THE SURJECTIVITY OF THE COMBINATORIAL LAPLACIAN ON INFINITE GRAPHS

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ABSTRACT. Applying a well-known theorem due to Eidelheit, we give a short proof of the surjectivity of the combinatorial Laplacian on connected locally finite undirected simplicial graph G with countably infinite vertex set V established in [1]. In fact, we show that every linear operator on \mathbb{K}^V which has finite hopping range and satisfies the pointwise maximum principle is surjective.

1. INTRODUCTION

In [1] Ceccherini-Silberstein, Coornaert, and Dodziuk showed that on a connected locally finite simplicial undirected graph G with a countably infinite vertex set V the combinatorial Laplacian on the real valued functions on V , $\Delta_G : \mathbb{R}^V \rightarrow \mathbb{R}^V$ defined by

$$\forall v \in V : \Delta_G f(v) = f(v) - \frac{1}{\deg(v)} \sum_{v \sim w} f(w),$$

is surjective. Two vertices $v, w \in V$ of G are adjacent, $v \sim w$, if (v, w) is an edge of G . Recall that a graph is locally finite if for every vertex v of G the number $\deg(v)$ of adjacent vertices is finite and that G is connected if for any pair of different vertices v, w there is a finite number of edges $(v_0, v_1), \dots, (v_{n-1}, v_n)$ of G with $v \in \{v_0, v_n\}$ and $w \in \{v_0, v_n\}$. Finally, G is simplicial if it does not have any loops, i.e. (v, v) is not an edge of G for any vertex v , and G is undirected if (w, v) is an edge of G whenever (v, w) is an edge of G .

In [1] the surjectivity of the Laplacian was proved by a Mittag-Leffler argument and an application of the (pointwise) maximum principle for Δ_G . There is a vast amount of literature dealing with a systematic study of the Mittag-Leffler procedure and its applications to a wide range of surjectivity problems, see e.g. [4] and the references therein. As noted in [1], the prove of surjectivity of Δ_G extends to more general operators, e.g. to operators of the form $\Delta_G + \lambda$, where $\lambda : V \rightarrow [0, \infty)$.

The aim of this note is to give a very short proof of the following generalisation of the above result.

Theorem 1. *Let G be a locally finite connected graph (which may be directed or undirected) with countably infinite vertex set V . Every linear operator $A : \mathbb{K}^V \rightarrow \mathbb{K}^V$ which has finite hopping range and satisfies the pointwise maximum principle is surjective.*

Note that Δ_G has finite hopping range and satisfies the pointwise maximum principle. (For a precise definition of finite hopping range and the pointwise maximum principle see section 2.) The proof of the above theorem relies on a well-known result due to Eidelheit.

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2. PROOF OF THEOREM 1

We equip $\mathbb{K}^{\mathbb{N}}$ with its usual Fréchet space topology, i.e. the locally convex topology defined by the increasing fundamental system of seminorms $(p_k)_{k \in \mathbb{N}}$ given by

$$(1) \quad \forall f = (f_j)_{j \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : p_k(f) = \sum_{j=1}^k |f_j|.$$

As usual, we denote this Fréchet space by ω . The dual space ω' of ω is given by the space of finitely supported sequences φ . For $j \in \mathbb{N}$ we set $\pi_j : \omega \rightarrow \mathbb{K}, (f_m)_{m \in \mathbb{N}} \mapsto f_j$ so that $\varphi = \text{span}\{\pi_j; j \in \mathbb{N}\}$.

If E is any Fréchet space and $A : E \rightarrow \omega$ linear and continuous we set $A_j := \pi_j \circ A, j \in \mathbb{N}$, so that $A_j \in E'$. A straight forward calculation gives that the transpose $A^t : \varphi \rightarrow E'$ is given by $A^t(y) = \sum_{j=1}^{\infty} y_j A_j$, where only finitely many y_j 's do not vanish. Thus, the linear independence of $(A_j)_{j \in \mathbb{N}}$ is equivalent to the injectivity of A^t . By the Hahn-Banach Theorem, the later is equivalent to A having dense range. We recall the following theorem due to Eidelheit (see e.g. [2] or [3, Theorem 26.27]).

Eidelheit's Theorem *Let E be a Fréchet space, $(p_k)_{k \in \mathbb{N}}$ be an increasing fundamental system of seminorms on E and let $A : E \rightarrow \omega$ be linear and continuous. Then, A is surjective if, and only if, for $(A_j)_{j \in \mathbb{N}}$ defined as above the following conditions are satisfied.*

- i) $(A_j)_{j \in \mathbb{N}}$ is linearly independent.
- ii) For every $k \in \mathbb{N}$

$$\dim(\{\phi \in E' : \exists c > 0 : |\phi| \leq c p_k\} \cap \text{span}\{A_j; j \in \mathbb{N}\}) < \infty.$$

For $E = \omega$ with the increasing fundamental system of seminorms $(p_k)_{k \in \mathbb{N}}$ given by (1), it is immediate that for $y \in \varphi$ there is $c > 0$ with $|\langle y, f \rangle| \leq c p_k(f)$ for all $f \in \omega$ precisely when $y \in \varphi_k := \{x = (x_j)_{j \in \mathbb{N}} \in \omega; x_j = 0 \text{ for all } j > k\}$. Since φ_k is obviously finite dimensional, by Eidelheit's Theorem, a linear continuous operator $A : \omega \rightarrow \omega$ is surjective if and only if $(A_j)_{j \in \mathbb{N}}$ is linearly independent.

Now, let G be a locally finite connected graph with countably infinite vertex set V . For each $v \in V$ and $n \in \mathbb{N}$ we define $U_n(v)$ to be the union of $\{v\}$ with the set of all endpoints of paths in G starting in v and of length not exceeding n together with the set of starting points of paths in G ending in v and of length not exceeding n .

Definition 1. Let G be a graph with vertex set V and let $A : \mathbb{K}^V \rightarrow \mathbb{K}^V$ be linear.

- i) A has *finite hopping range* if for every $v \in V$ there is $n \in \mathbb{N}$ such that the following implication holds

$$\forall f, g \in \mathbb{K}^V : f|_{U_n(v)} = g|_{U_n(v)} \Rightarrow A(f)(v) = A(g)(v).$$

- ii) A satisfies the *pointwise maximum principle* if for every $v \in V$ there is $n \in \mathbb{N}$ such that for each $f \in \mathbb{K}^V$ with $A(f)(v) = 0$ the implication

$$|f(v)| = \max_{U_n(v)} |f(w)| \Rightarrow \forall w \in U_n(v) : |f(w)| = |f(v)|$$

holds.

Enumeration of the vertices $V = \{v_k; k \in \mathbb{N}\}$ of G clearly gives an isomorphism of \mathbb{K}^V onto $\mathbb{K}^{\mathbb{N}}$. In order to keep notation simple, for a linear mapping $A : \mathbb{K}^V \rightarrow \mathbb{K}^V$ we denote by A also the linear operator on $\mathbb{K}^{\mathbb{N}}$ induced by the canonical isomorphism between \mathbb{K}^V and $\mathbb{K}^{\mathbb{N}}$. Since the linear operator $A : \omega \rightarrow \omega$ is continuous if and only if $A_j \in \varphi$ for all $j \in \mathbb{N}$, using that G is connected and locally finite, a

straight forward calculation shows that $A : \omega \rightarrow \omega$ is continuous if and only if the inducing $A : \mathbb{K}^V \rightarrow \mathbb{K}^V$ has finite hopping range.

Proposition 2. *Let G be a locally finite connected graph with countably infinite vertex set V and let $A : \mathbb{K}^V \rightarrow \mathbb{K}^V$ be linear. If A has finite hopping range and satisfies the pointwise maximum principle, then $(A_j)_{j \in \mathbb{N}}$ is a linearly independent sequence of continuous linear forms on ω .*

Proof. As pointed out above, the claim is equivalent to the injectivity of A^t . For $k \in \mathbb{N}$ we define

$$M_k : \omega \rightarrow \omega, (f_j)_{j \in \mathbb{N}} \mapsto (f_1, \dots, f_k, 0, \dots).$$

Then M_k is a continuous linear operator on ω with $M_k^t = M_{k|\varphi}$.

Fix $y \in \varphi$ with $A^t(y) = 0$. Then there is $k \in \mathbb{N}$ with $y \in \varphi_k$ and

$$\begin{aligned} \forall f \in \varphi_k : 0 &= \langle y, A(f) \rangle = \langle M_k^t(y), (A_{|\varphi_k}(f)) \rangle = \langle y, (M_k \circ A_{|\varphi_k})(f) \rangle \\ &= \langle (M_k \circ A_{|\varphi_k})^{t_k}(y), f \rangle_k, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_k$ denotes the duality bracket between φ'_k and φ_k and $(M_k \circ A_{|\varphi_k})^{t_k}$ the transpose of

$$M_k \circ A_{|\varphi_k} : \varphi_k \rightarrow \varphi_k$$

with respect to this duality. From the above we conclude that $(M_k \circ A_{|\varphi_k})^{t_k}(y) = 0$ so that $y = 0$ if we can show that $(M_k \circ A_{|\varphi_k})^{t_k}$ is injective. Since φ_k is finite dimensional, the later holds precisely when $M_k \circ A_{|\varphi_k}$ is injective, which will be proved as in [1] by the pointwise maximum principle for A .

For $j \in \mathbb{N}$ we define $N(j) = \{l \in \mathbb{N}; v_l \in U_n(v_j)\}$, where n is chosen as in Definition 1 ii) for $v = v_j$. Let $f \in \varphi_k$ satisfy $(M_k \circ A)(f) = 0$ and let $1 \leq k_0 \leq k$ be such that $|f_{k_0}| = \max\{|f_j| : j \in \mathbb{N}\} =: M$. By $0 = (M_k \circ A)(f)(k_0) = A(f)(k_0)$ and the pointwise maximum principle we conclude $|f_j| = M$ for all $j \in N(k_0)$. If there is $j \in N(k_0)$ with $j > k$ then $M = 0$, since f vanishes in $\{k+1, \dots\}$. If $N(k_0) \subseteq \{1, \dots, k\}$ it follows again from the maximum principle that $|f_j| = M$ for all $j \in \cup_{l \in N(k_0)} N(l)$. Using that G is connected, a finite numbers of iterations of this process finally yields that $|f_j| = M$ for some $j \in \{k+1, \dots\}$, where f vanishes so that $M = 0$, i.e. $f = 0$. \square

Proof of Theorem 1. Theorem 1 follows now immediately from the considerations following Eidelheit's Theorem and the above proposition. \square

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